Closed Orbit Part 2: Physical Significance

Kai Hock

Cockcroft Institute / Liverpool University

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In the note on closed orbit [1], I mentioned that in the presence of imperfections, an electron would follow a new path, called the closed orbit. Why can't it follow another path? Of course it can. If the electron receives a large force, it can even leave the accelerator. What I would like to show in this note is:

If the electron is near the closed orbit, and the radiation loss is small, then the path of electron would get closer and closer to the closed orbit.

To deomnstrate this, we start by noting that:

1. If an electron happens to be on one point of the closed orbit - with the exact displacements and momenta - then it follows the closed orbit. This is because its path is determined by the same transfer matrices that give the closed orbit.

2. If an electron happens not to be on any point in the closed orbit, then it does not follow the closed orbit. It does not even intersect the closed orbit, otherwise the previous point applies and it would follow the closed orbit.

We want to show that an electron not on the closed orbit would still approach the closed orbit after some time. In order to demonstrate this, the transfer matrices must first be linearised around the closed orbit. This means:

1. finding a new, one turn transformation matrix for which the close orbit is now the origin

2. this matrix would give the correct co-ordinates after one turn for small deviations from the closed orbit.

Denote a point on the closed orbit at position s by $(x_0, x_0', y_0, y_0', z_0, \delta_0, 1)$.

Denote the deviation from this point of an electron at s by $(x, x', y, y', z, \delta, 1)$.

With respect to original reference path, the actual co-ordinates of the electron is $(x_0 + x, x_0' + x', y_0 + y, y_0' + y', z_0 + z, \delta_0 + \delta, 1)$.

Denote the co-ordinates after one turn by $(x_0 + x_2, x_0' + x_2', y_0 + y_2, y_0' + y_2', z_0 + z_2, \delta_0 + \delta_2, 1)$.

Start with the example of a horizontal bending magnet [5]. The 7x7

transfer matrix in the original reference path co-ordinates is given by

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{1}$$

In terms of the closed orbit and the deviation, it can be written as:

$$\begin{pmatrix} x_0 + x_2 \\ x_0' + x_2' \\ y_0 + y_2 \\ y_0' + y_2' \\ z_0 + z_2 \\ \delta_0 + \delta_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 + x \\ x_0' + x' \\ y_0 + y \\ y_0' + y' \\ z_0 + z \\ \delta_0 + \delta \\ 1 \end{pmatrix} \tag{2}$$

For the point at s on the closed orbit itself, the initial and final positions are

the same, by definition of the closed orbit:

$$\begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{3}$$

In order to linearise around the closed orbit, we subtract one matrix equation from the the other. To make it easier, we expand each matrix equation into the seven equations. As a starting example, we look at the second of the seven equations in eq. (2):

$$x_0' = x_0' + m_{26}\delta_0 + m_{27} \tag{4}$$

and in eq. (3):

$$x_0' + x_2' = (x_0' + x) + m_{26}(\delta_0 + \delta) + m_{27} \tag{5}$$

Subtracting, we get

$$x_2' = x + m_{26}\delta \tag{6}$$

Repeating this subtraction for all the equations in eqs. (2) and (3), we

get

$$\begin{pmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \\ z_2 \\ \delta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 0 \end{pmatrix} \tag{7}$$

Notice that the last row and last column of the matrix are all zero. So the matrix has contracted to a 6x6 matrix. The equation can be written in terms of the 6x6 matrix:

$$\begin{pmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \\ z_2 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \end{pmatrix} \tag{8}$$

Note that the displacement vectors has only 6 elements now.

In the same way, the linearised 6x6 matrices for all the other ring elements can be obtained.

There is a slight complication for sextupole because the matrix elements

depend on x and y [2]:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & m_{46} & m_{47} \\ m_{51} & 0 & m_{53} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{9}$$

This can be expanded into 7 equations as before. The 2nd equation is:

$$x_1' = x_0' + m_{26}\delta_0 + m_{27} \tag{10}$$

Substituting the expressions from [2] gives:

$$x_1' = x_0' + \frac{1}{2}\lambda(x_0^2 + y_0^2)\delta_0 - \frac{1}{2}\lambda(x_0^2 + y_0^2) \tag{11}$$

In terms of the co-ordinates for the closed orbit and the deviation, this becomes:

$$x_0' = x_0' + \frac{1}{2}\lambda(x_0^2 - y_0^2)\delta_0 - \frac{1}{2}\lambda(x_0^2 - y_0^2) \tag{12}$$

and

$$x_0' + x_2' = (x_0' + x') + \frac{1}{2}\lambda[(x_0+x)^2 - (y_0+y)^2](\delta_0+\delta) - \frac{1}{2}\lambda[(x_0+x)^2 - (y_0+y)^2] \tag{13}$$

subtract - neglect higher order such as $x\delta$, $x^2\delta$, x^2, $y\delta$, $y^2\delta$, and y^2, we get:

$$x_2' = x' + \frac{1}{2}\lambda(x_0^2 - y_0^2)\delta - \lambda x_0 x(1-\delta_0) + \lambda y_0 y(1-\delta_0) \tag{14}$$

In [3], the factor of $(1-\delta_0)$ is apparently neglected, giving

$$x_2' = x' + \frac{1}{2}\lambda(x_0^2 - y_0^2)\delta - \lambda x_0 x + \lambda y_0 y \tag{15}$$

Likewise, the 4th equation from eq. (9) is:

$$y_1' = y_0' + m_{46}\delta_0 + m_{47} \tag{16}$$

Substituting the expressions from [2] gives:

$$y_1' = y_0' - \lambda x_0 y_0 \delta_0 + \lambda x_0 y_0 \tag{17}$$

In terms of the co-ordinates forthe closed orbit and the deviation, this becomes:

$$y_0' = y_0' - \lambda x_0 y_0 \delta_0 + \lambda x_0 y_0 \tag{18}$$

and

$$y_0' + y_2' = (y_0' + y') - \lambda(x_0 + x)(y_0 + y)(\delta_0 + \delta) + \lambda(x_0 + x)(y_0 + y) \tag{19}$$

subtracting and neglecting higher order therms like $x\delta$, $xy\delta$, xy, and $y\delta$ gives

$$y_2' = y' - \lambda x_0 y_0 \delta + \lambda x_0 y(1+\delta_0) + \lambda y_0 x(1+\delta_0) \tag{20}$$

Again, in [3], the factor of $(1-\delta_0)$ is apparently neglected, giving

$$y_2' = y' - \lambda x_0 y_0 \delta + \lambda x_0 y + \lambda y_0 x \tag{21}$$

Repeating the above steps for the 5th equation from eq. (9) gives:

$$z_2 = -\frac{3}{2}\lambda(x_0^2 - y_0^2)x + 3\lambda x_0 y_0 y + z \tag{22}$$

Note that in [3], the factor of 3 is not there.

Combining the above results, the linearised sextupole transfer matrix is:

$$\begin{pmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \\ z_2 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda x_0(1-\delta_0) & 1 & \lambda y_0(1-\delta_0) & 0 & 0 & \frac{1}{2}\lambda(x_0^2 - y_0^2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda y_0(1-\delta_0) & 0 & \lambda x_0(1-\delta_0) & 1 & 0 & -\lambda x_0 y_0 \\ -\frac{3}{2}\lambda(x_0^2 - y_0^2) & 0 & 3\lambda x_0 y_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \tag{23}$$

Note difference from expression in [3]:

$$\begin{pmatrix} x_2 \\ x_2' \\ y_2 \\ y_2' \\ z_2 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda x_0 & 1 & \lambda y_0 & 0 & 0 & \frac{1}{2}\lambda(x_0^2 - y_0^2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda y_0 & 0 & \lambda x_0 & 1 & 0 & -\lambda x_0 y_0 \\ -\frac{1}{2}\lambda(x_0^2 - y_0^2) & 0 & \lambda x_0 y_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \tag{24}$$

After obtaining all the 6x6 linearised matrices, the next step is to multiply them around the ring to get the one turn 6x6 transformation matrix at $T(s)$.

Then we solve the eigenvalue problem:

$$T(s)E(s) = \lambda E(S) \tag{25}$$

To see how this is related to the deviation from the closed orbit, write the eigenvalues as:

$$\lambda_k = \exp(-\alpha_k + \pm i2\pi\nu_k) \tag{26}$$

for the eigenvectors $E_k(s)$ and $E_k(s)^*$, for $k = I, II, III$. The deviation from the closed orbit can be expressed in terms of the eigenvectors:

$$X(s) = \sum_k c_k E_k(s) + c_k^* E_k(s)^* \tag{27}$$

After one turn, this would become

$$X(s + 2\pi R) = \sum_k e^{-\alpha_k}[c_k E_k(s) e^{+i2\pi\nu_k} + c_k^* E_k(s)^* e^{-i2\pi\nu_k}] \tag{28}$$

So if all three of the α_k are positive, the deviation would get smaller and smaller with every turn around the ring, and the electron would approach the closed orbit.

We now need to understand why or how α_k could positive.

It turns out that the damping depends on both the radiation loss at magnets, and energy gain in cavities. Consider energy gain u in a cavity. This causes the gradients x' and y' to decrease by a factor of $1 - u/E_0$. The reason is illustrated in fig. 1.

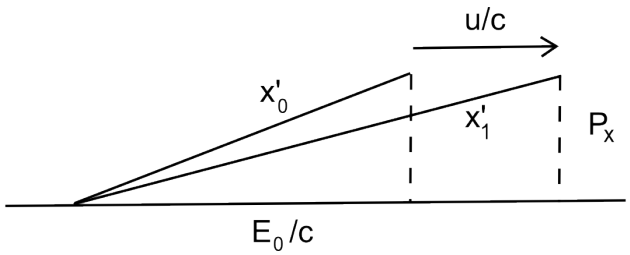

Figure 1: Damping effect of the cavity.

Let $P_0 = E_0/c$ be the full electron momentum. Let P_x be the actual transverse momentum. Then the gradient x' is given by

$$x'_0 = \frac{P_x}{E_0/c} \tag{29}$$

Figure 1 shows that if there is an energy gain of u, the gradient becomes

$$x'_1 = \frac{P_x}{E_0/c + u/c} \tag{30}$$

Since the energy gain in the cavity u is small compared to the electron energy E_0, a first order expansion gives:

$$x'_1 \approx \frac{P_x}{E_0/c}\left(1 - \frac{u}{E_0}\right) = \left(1 - \frac{u}{E_0}\right) x'_0 \tag{31}$$

This transformation from x'_0 to x'_1 can be included in the 6x6 matrix by changing the value of m_{22} from 1 to

$$m_{22} = 1 - \frac{u}{E_0} \tag{32}$$

Applying the same reasoning to the gradient y' gives

$$m_{44} = 1 - \frac{u}{E_0} \tag{33}$$

Next, we consider the additional radiation loss in a magnet, as a result of the deviation δ from the closed orbit.

Previously, when the 7x7 matrix is used, the radiation loss is included in the 7th column. E.g. m_{67} for the bending magnet [5] is used to include radiation loss in the value of δ_1.

For the 6x6 matrix, this can be done by modifying the value of the diagonal element m_{66} for a magnet. The required change is sketched out in [4]. I shall fill in some details here.

Following the approach of [4], the radiation loss is writtien in this form:

$$P = \frac{2}{3}\left(\frac{e^4B^2}{m^2c^3}\right)\left(\frac{E}{mc^2}\right)^2 \tag{34}$$

where m is the electron rest mass m_e. This is derived from Lienard's formula [6]

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \tag{35}$$

where q is the electron charge e, a is the acceleration. Equation (34) derived from Lienard's formula using the following:

$$E = \gamma mc^2 \text{ (energy mass relation)} \tag{36}$$

$$\gamma ma = Bec \text{ (Newton's second law and magnetic force)} \tag{37}$$

We need to apply eq. (34) to a magnet element. In this element, the magnetic field B is fixed. We need to find the additional radiation loss when

energy E deviates from its value on the closed orbit by δ For the relative amount δ, the actual amount is $E\delta$.

The radiation loss on the closed orbit at the magnet elment is $\epsilon = Pt$, where t is the time taken through the element, This amount has already be included in the 7x7 matrix when calculating the closed orbit When the energy is changed by $E\delta$ from the closed orbit, there is additional radiation loss. We can find this in terms of ϵ.

Notice from eq. (34) that P is proportional to E^2. If the energy changes by a small amount ΔE, P changes to $P(1 + 2\Delta E/E)$. So when the energy change is $E\delta$, this become $\epsilon(1 + 2\delta)$. This gives a relative change of $\epsilon(1 + 2\delta)/E_0$ with respect to the reference energy E_0. This energy loss has to be subtracted from δ.

Split the energy loss into two parts: ϵ/E_0 and $2\epsilon\delta/E_0$. The first part, ϵ/E_0, has already been subtracted during the closed orbit calculation. E.g. for the bending magnet, it is the term m_{67} [5]. The other part, $2\epsilon\delta/E_0$, must now be subtracted from δ for the deviation from closed orbit.

That is, δ must be changed to $\delta - 2\epsilon\delta/E_0$. This can be done by modifying the element m_{66} in the 6x6 transfer matrix for the magnet. Lets write $\delta - 2\epsilon\delta/E_0$ as $(1 - 2\epsilon/E_0)\delta$. So when the transfer matrix is applied to the deviation vector, we need δ to change to $(1 - 2\epsilon/E_0)\delta$. We can achieve this if we replace the value of 1 for m_{66} by $(1 - 2\epsilon/E_0)$.

Summarising the results so far- Of the 6 diagonal elements in the 6x6 transfer matrix, 3 needs to be modified:

In a cavity, for the co-ordinates x' and y',

$$m_{22} = 1 - u/E_0 \tag{38}$$

$$m_{44} = 1 - u/E_0 \tag{39}$$

In a magnet, for the co-ordinates δ,

$$m_{66} = 1 - 2\epsilon/E_0 \tag{40}$$

These expression give us a way to relate the energy gain and loss to the eigenvalues, and then to the damping rates.

The product of the eigenvalues is equal to the determinant of the matrix. For an infinitesimal element of the ring, the determinant can be calculated using just the above elements. An infinitesimal element means a very short element in the ring. This can be a hypothetical fraction of a magnet, a fraction of a drift space, or a fraction of cavity. As explained in [4], "the only terms in the determinant which will be first order in the length of the element will be due to the diagonal terms of the matrix."

To see that this is so, an inspection of the off diagonal elements for the transfer matrices given in Table 1 of [3] would show that they are indeed all to first order in length l. This is less obvious for the matrix elements for the

rf cavity, because it is given in terms of the potential difference $\hat{V}$ across the whole cavity. However, if we imagine dividing the cavity up into fractions, then $\hat{V}$ would also change according to the length of each fraction.

A transfer matrix for an infinitesimal ring element, therefore, has the property that all the off diagonal matrix elements are first order in length. The diagonal matrix elements are 1 except m_{22}, m_{44} and m_{66} above. For these matrix elements, the difference from 1 comes from the energy gain or loss, which are also proportional to the length l of the ring element.

To see that the determinant will depend to first order on the diagonal terms only, consider the following simple example:

$$A = \begin{pmatrix} 1 + k_1 a & k_2 a \\ k_3 a & 1 + k_4 a \end{pmatrix} \tag{41}$$

where the terms in a are all much smaller than 1. The determinant is

$$det(A) = (1 + k_1 a)(1 + k_4 a) - k_2 k_3 a^2 \tag{42}$$

$$= 1 + k_1 a + k_4 a + k_1 k_4 a^2 - k_2 k_3 a^2 \tag{43}$$

To first order in a, the determinant is given by $1 + k_1 a + k_4 a$. These include just the terms in the off diagonal elements, as pointed out in [4].

Likewise, for the 6x6 transfer matrix, the determinant would be given by $1 + \sum \delta_{nn}$, where δ_{nn} is the difference of the n^{th} diagonal element from 1. For

a cavity, where

$$m_{22} = 1 - \frac{u}{E_0} \tag{44}$$

$$m_{44} = 1 - \frac{u}{E_0} \tag{45}$$

the determinant would be $1 - 2u/E_0$. For a magnet, where

$$m_{66} = 1 - \frac{2\epsilon}{E_0} \tag{46}$$

the determinant would be $1 - 2\epsilon/E_0$.

Next, we find the determinant of the matrix for one revolution. This is obtained by multiplying all the transfer matrices of the elements in the ring. We use the property that the determinant of a product is equal to the product of the determinants:

$$\det(AB) = \det(A)\det(B) \tag{47}$$

Consider the product of a magnet transfer matrix and a cavity transfer matrix. Using the determinants obtained above, the determinant of the product is, to first order,

$$\left(1 - \frac{2u}{E_0}\right)\left(1 - \frac{2\epsilon}{E_0}\right) \approx 1 - \frac{2u}{E_0} - \frac{2\epsilon}{E_0} \tag{48}$$

i.e. we just have to add 1 to all of the δ_{nn} in both matrices. This is readily extended to the product of all the matrices in the ring:

$$\det(T(s)) \approx 1 - \frac{2\sum u}{E_0} - \frac{2\sum \epsilon}{E_0} \tag{49}$$

where $\sum u$ is the total energy gain U_0 around the ring, and $\sum \epsilon$ is the total radiation loss. For the equilibrium condition, when the electron follows a steady orbit - the total energy gain is equal to the total radiation loss. Then

$$\det(T(s)) \approx 1 - \frac{4U_0}{E_0} \tag{50}$$

To relate to the eigenvalues, we find the determinant from eq. (26):

$$\det(T(s)) = \exp[-2(\alpha_I + \alpha_{II} + \alpha_{III})] \tag{51}$$

For small radiation loss, we expect damping to be small, so that

$$\det(T(s)) \approx 1 - 2(\alpha_I + \alpha_{II} + \alpha_{III}) \tag{52}$$

Comparing eqs. (50) and (52),

$$\alpha_I + \alpha_{II} + \alpha_{III} = \frac{2U_0}{E_0} \tag{53}$$

This is the Robinson's sum rule.

In the limit of no closed orbit distortion and no coupling (the motion in one axis does not affect motion in another axis), the transfer matrice can be diagonalised into three 2x2 matrices, one for each axis. The above reasoning can be applied to each direction separately. The x and y directions depend only on magnet elements:

$$\alpha_x = \alpha_y = \frac{U_0}{2E_0} \tag{54}$$

The z direction depends only on rf cavities:

$$\alpha_z = \frac{2U_0}{E_0} \tag{55}$$

So for non zero radiation loss U_0, the damping rate for all directions is positive, and an oscillating electron would fall toward the reference path.

However, even in the ideal case of no misalignment, there would be coupling. For instance, energy gain in a cavity would lead to decrease in δ, which would lead in turn to a change in x' through the next magnet. This means coupling between the x and z directions.

However, for small radiation loss, and for small misalignments, we may expect that the values of the damping rates would not deviate too far from α_x, α_y and α_z. Then the electron should approach the closed orbit after a number of turns.

The Robinson's sum rule tells us that the sum of the damping rates is positive. It does not say whether a particular damping rate has to be positive. However, the damping rates can be calculated from the one turn transformation matrix. This provides a way to design the ring elements to make all three damping rates positive. The existence of operational rings where stable beams are achieved shows that this is possible. In such cases, the electron would indeed approach the closed orbit.

References

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[4] K. Robinson, Physical Review, **111**, 373 (1958).

[5] http://hep.ph.liv.ac.uk/~hock/Damping_Ring/bend_magnet.html

[6] http://hep.ph.liv.ac.uk/~hock/Damping_Ring/radiation.html